

Fun with cascades

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Abstract. This paper is, in a certain sense, a supplement of S. Dolecki's paper *Multisequences* [Quaest. Math. **29** (2006), 239–277]. We describe some operations on cascades, together with their influence on a contour and sometimes on other notions connected with cascades. We illustrate these operations by a sketch of their use in some proofs of the results published last years concerning (ultra)filters. The paper contains also some new results on subsequential filters, answering a question from [Garcia-Ferreira S., Uzcáteui C.: *Subsequential filters*, Toppology. Appl. **156** (2009), 2949–2959]. To make the paper self-contained, we repeat all necessary definitions and re-describe some properties.

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1. Preliminaries

Monotone sequential cascades were introduced by S. Dolecki and F. Mynard in [5] to describe topological sequential spaces. Recently they were also used by S. Garcia-Ferreira and C. Uzcáteui in [10, 11] as a tool to investigate subsequential spaces. The reader interested in topological/convergence use of cascades is invited to look also at [6, 7]. Here we focus on pure set-theoretical aspects of cascades and, in fact, on infinite combinatorics used to this end. For brevity we omit proofs of quoted theorems and present only sketches of those of them, which show the typical way how to work with cascades.

A *cascade* is a tree V , ordered by “ \sqsubseteq ”, without infinite branches and with the least element \emptyset_V . A cascade is *sequential* if for each non-maximal element v of V ($v \in V \setminus \max V$) the set v^{+V} of immediate successors of v (in V) is countably infinite.

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We say that v is a *predecessor* of v' (we write $v = \text{pred}(v')$ or $v = v'^{-V}$) if $v' \in v^{+V}$. We write v^+ (v^-) instead of v^{+V} (v^{-V}) if it is known in which cascade the successors (predecessor) of v is considered. As a convention, since $\max V$ is a countably infinite set, we think about $\max V$ as of a copy (or of a subset) of ω .

The *rank* of $v \in V$ ($r_V(v)$ or $r(v)$) is defined inductively as follows: $r(v) = 0$ if $v \in \max V$, otherwise $r(v)$ is the least ordinal greater than the ranks of all immediate successor of v . The rank $r(V)$ of the cascade V is, by definition, the rank of \emptyset_V . Note that for each countable ordinal α there is a (monotone – see next paragraph) sequential cascade of rank α .

The cascade V is said to be *monotone* if it is possible to order all sets v^+ (for $v \in V \setminus \max V$) in ω -sequences $(v_n)_{n < \omega}$ so that for each $v \in V \setminus \max V$ the sequence $(r(v_n)_{n < \omega})$ is nondecreasing. In this case we fix such an order on V without indication. Equivalently, a cascade is monotone if for each $v \in V \setminus \emptyset_V$ the set $\{v' \in (\text{pred}(v))^+ : r(v') < r(v)\}$ is finite. Then we can introduce the lexicographic order $<_{lex}$ on V in the following way: $v' <_{lex} v''$ if $v' \sqsupset v''$ or if there exist $\tilde{v}' \sqsubseteq v'$, $\tilde{v}'' \sqsubseteq v''$ and v such that $\tilde{v}' \in v^+$ and $\tilde{v}'' \in v^+$ and $\tilde{v}' = v_n$, $\tilde{v}'' = v_m$ and $n < m$. While we have fixed the lexicographic order on a cascade V , we can label elements of V by finite sequences of natural numbers of length $r(V)$ or less, by the function f which preserves the lexicographic order, $v_{f(v)}$ is the resulting name for an element of V , where $f(v)$ is the mentioned sequence. A function $f : V \rightarrow \omega^{<\omega}$ is defined inductively by: $f(\emptyset_V) = \emptyset$ and if $f(v)$ is already defined, then $f(v_n) = f(v) \frown n$ (where v_n is the n -th element of v^+). For $v \in V$ we denote by $v^{\uparrow V}$ a subcascade of V built by v and all successors of v . For $l \in \omega^{<\omega}$ by V_l we denote v_l^{\uparrow} and by $L_{\alpha,V}$ we understand $\{l \in \omega^{<\omega} : r_V(v_l) = \alpha\}$. It will be clear from the context whether v_n is the n -th element of v^+ , or v_n is the n -th element of \emptyset_V^+ .

To define filters related to cascades we need the following notion of confluence of cascades, which should be mentioned in Section *Operations*, but for the above sake is defined here. Let W be a cascade and let $\{V^w : w \in \max W\}$ be a set of pairwise disjoint cascades such that $V^w \cap W = \emptyset$ for each $w \in \max W$. The *confluence* of cascades V^w with respect to the cascade W (we write $W \uparrow V^w$) is defined as a cascade constructed by the identification of $w \in \max W$ with \emptyset_{V^w} and according to the following rules:

- (1) $\emptyset_W = \emptyset_{W \uparrow V^w}$;
- (2) if $w \in W \setminus \max W$, then $w^{+W \uparrow V^w} = w^{+W}$;
- (3) if $w \in V^{w_0}$, for a certain $w_0 \in \max W$, then $w^{+W \uparrow V^w} = w^{+V^{w_0}}$;
- (4) in each case we also assume that the order on the set of successors remains unchanged.

By $(n) \uparrow V_n$ we denote $W \uparrow V^w$ where W is a sequential cascade of rank 1. Each cascade V of rank ≥ 1 may be uniquely described as a confluence of cascades of rank less than $r(V)$ – simply $V = (n) \uparrow V_n$. As a convention, if u is a filter(-base) on $A \subset B$, then we identify u with the filter on B for which u is a filter-base.

If V is a monotone sequential cascade of rank 1 then the *contour* of V (in symbols $\int V$) is a filter on $\max V$ defined inductively: if $r(V) = 1$, then $\int V$ is a Fréchet filter on $\max V$ and if $V = (n) \uparrow V_n$, then $U \in \int V$ if and only if $U \cap \max V_n \in \int V_n$ for almost all $n < \omega$. If u is a filter such that $u = \int V$ for some monotone sequential

cascade V , then we say that u is a monotone sequential contour. We assume (if not indicated otherwise) that all filters in this paper are considered on ω .

The contour of a cascade is strongly related to the following operation defined for a family of filters $\mathbb{U} = \{u_s\}_{s \in S}$ on a fixed set and for a filter u on S . The operation is probably best known as a Frolik sum, but also as a limit of filters and as a contour of filters, and it is denoted by $\sum_u u_s$, $\lim_u u_s$ and $\int_u u_s$, respectively. Here we keep the last notion, since we did it in previous papers.

$$\int_p \mathbb{U} = \int_p u_s = \bigcup_{P \in p} \bigcap_{s \in P} u_s.$$

Similar constructions were used by several authors ([3, 9, 12, 13]).

In the paper we will mainly work on four types of filters – monotone sequential contours, elements of the P-hierarchy, ordinal ultrafilters and subsequential filters.

J.E. Baumgartner in ([1]) introduced a notion of ordinal ultrafilters, hierarchized in an ω_1 -sequence of classes of ultrafilters. We say that u is a J_α -ultrafilter (on ω) if for each function $f : \omega \rightarrow \omega_1$ there is $U \in u$ such that $\text{ot}(f[U]) < \alpha$, where $\text{ot}(\cdot)$ denotes the order type. An ultrafilter u on ω is a *strict* J_α -ultrafilter if u is a J_α -ultrafilter and u is not J_β -ultrafilter for any $\beta < \alpha$. For additional information about ordinal ultrafilters a look at [1, 2, 16, 18, 19, 20, 21, 22] is recommended.

Another way of classifying ultrafilters into ω_1 -sequence of classes is P-hierarchy, which has been defined in [20] as follows: $u \in \mathcal{P}_\alpha$ if there is no monotone sequential contour \mathcal{V}_α of rank α such that $\mathcal{V}_\alpha \subset u$, and for each β in the range $1 \leq \beta < \alpha$ there exists a monotone sequential contour \mathcal{V}_β of rank β such that $\mathcal{V}_\beta \subset u$. Moreover, if for each $\alpha < \omega_1$ there exists a monotone sequential contour \mathcal{V}_α of rank α such that $\mathcal{V}_\alpha \subset u$, then we write $u \in \mathcal{P}_{\omega_1}$. For additional information about P-hierarchy see [21].

In [10] we presented a following definition of *subsequential* filters introduced by N. Noble in [17]: A space X is said to be *sequential* if for each non-closed subset A of X there is a sequence $(x_n)_{n < \omega}$ in A that converges to a point in $X \setminus A$. A space is called *subsequential* if it can be embedded in a sequential space. If \mathcal{F} is a free filter on ω , then the symbol $\xi(\mathcal{F})$ stands for the space whose underlying set is $\omega \cup \{\mathcal{F}\}$, where ω is a discrete subset and a basic neighbourhood of \mathcal{F} is of the form $F \cup \mathcal{F}$, where $F \in \mathcal{F}$. A free filter \mathcal{F} is called *subsequential* if the space $\xi(\mathcal{F})$ is subsequential.

If for some monotone sequential cascade V we restrict the lexicographic order of V to $\max V$, then we obtain a set order-isomorphic to an indecomposable ordinal number, recall that ordinal α is indecomposable if $\alpha = \omega^\beta$ for some ordinal β . Similar relation was introduced first by Dolecki and Watson in [7], then slightly changed to the version presented by the author. This relation lets us use cascades in the investigation of ordinal ultrafilters. Formally for a monotone sequential cascade V by f_V we denote each *lexicographic order respecting function* $\max V \rightarrow \omega_1$, i.e., such a function that $v' <_{lex} v''$ if and only if $f_V(v') < f_V(v'')$ for each $v', v'' \in \max V$. If $f : \omega \rightarrow \omega_1$ and $f = f_V$ for some monotone sequential cascade V (i.e., by convention mentioned before, there is a bijection $g : \max V \rightarrow \omega$ such that $f \circ g = f_V$), then we say that V corresponds with the order of f .

2. Operations

2.1. Confluence \approx contour operation

The confluence, together with a contour operation (as we see these two are twins for sequential cascades/contours), are probably the most important operations on cascades and fruitful on filters. The definition has been given above, here we give a relation between them:

Let W be a cascade and let $\{V^w : w \in \max W\}$ be a set of pairwise disjoint cascades such that $V^w \cap W = \emptyset$ for each $w \in \max W$. Then

$$\int (W \leftarrow V^w) = \int_f W \int V^w.$$

Its typical use is for instance the proof of the fact that there are free ultrafilters which are not P-points. But what is especially interesting: although this operation increases the “level of complication” of a filter, it does not increase it too much. We will show it in three cases - of ordinal ultrafilters, of P-hierarchy and of subsequential contours.

2.1.1. Ordinal ultrafilters

Dolecki and Watson proved in [7] that if we look on maximal elements of cascades as at the ordered sets, then confluence gives as an ordered sum of order types, formally

Theorem 2.1 ([7]).

- 1) Let W be a cascade and let $\{V^w : w \in \max W\}$ be a set of pairwise disjoint cascades such that $V^w \cap W = \emptyset$ for each $w \in \max W$. Also, let f_{V^w}, f_W be order respecting functions. Then

$$\text{ot}(f_{W \leftarrow V^w}(\max(W \leftarrow V^w))) = \sum_{\text{ot}(f_W)(\max W)} \text{ot}(f_{V^w}(\max V^w)),$$

where $\text{ot}(\cdot)$ denotes order type and $\sum_{(\cdot)}(\cdot)$ denotes ordered sum of ordinal numbers.

- 2) Let W be a monotone sequential cascade and let f_W be an order respecting functions. Then $\text{ot}(f_W(\max W)) = \omega^r(W)$.

Since cascades define order functions of fixed rank, and since confluence does not increase them too much, we may expect the following result

Theorem 2.2 ([1, Theorem 4.2 restated in virtue of its proof]). Let $(\alpha_n)_{n < \omega}$ be a nondecreasing sequence of ordinals less than ω_1 , let $\alpha = \lim_{n < \omega}(\alpha_n)$. If (u_n) is a discrete sequence of strict $J_{\omega \alpha_n}$ -ultrafilters and u is a strict J_{ω^2} -ultrafilter, then $\int_u u_n$ is a strict $J_{\omega \alpha + 1}$ -ultrafilter.

Since Baumgartner also proved that the class of strict J_{ω^2} -ultrafilters is precisely the class of P-points [1, Theorem 4.1], the above theorem shows that existence of P-points implies non-emptiness of classes of ordinal ultrafilters of successor indices (and it was the original claim of Theorem 2.2.)

2.1.2. P-hierarchy

In the sequel we will use the following notation: for an ordinal α , by $(-1 + \alpha)$ we will understand α if $\alpha \geq \omega$ and $\alpha - 1$ if $\alpha < \omega$.

Theorem 2.3 ([20, Theorem 2.5]). *Let $(\alpha_n)_{n < \omega}$ be a nondecreasing sequence of ordinals less than ω_1 , let $\alpha = \lim_{n < \omega} (\alpha_n)$ and let $1 < \beta < \omega_1$. If $u_n \in \mathcal{P}_{\alpha_n}$ is a discrete sequence of ultrafilters and $u \in \mathcal{P}_\beta$, then $\int_u u_n \in \mathcal{P}_{\alpha+(-1+\beta)}$.*

Since we also proved that the class of \mathcal{P}_2 -ultrafilters is precisely the class of P-points [20, Proposition 2.1], the above theorem shows that the existence of P-points implies non-emptiness of classes of P-hierarchy of successor indices.

The previous sentence is a “homeomorphic copy” of the last sentence in subsection concerning ordinal ultrafilters, also main theorems of those subsections are similar and filters built in proofs of them are the same, so one can expect that classes of ordinal ultrafilters and of P-hierarchy may be the same. To show that it is not the case, we proved that

- 1) [20, Theorem 3.9] It is relatively consistent with ZEC that successor classes (with the same index) of ordinal ultrafilters and of P-hierarchy intersect, but are different;
- 2) [21, Theorem 6.5] (CH, and in unpublished yet article under weaker assumption) all classes of P-hierarchy are nonempty, while [22, Theorem 3.14] (ZFC) the class of strict J_{ω^ω} -ultrafilters is empty.

On the other hand, there are many other similarities between these hierarchies of ultrafilters, for more details see papers cited above.

2.1.3. Subsequential filters

Following [10] we define *degree of subsequentiality* of subsequential filters. Let X be a space and let $A \subset X$. We put $A^0 = A$ and for each ordinal number θ we define $A^\theta = \{x \in X : \exists (x_n)_{n < \omega} \subseteq A^\mu, (x_n) \rightarrow x\}$ if $\theta = \mu + 1$, and $A^\theta = \bigcup_{\mu < \theta} A^\mu$ if θ is a limit ordinal. It is known that a space X is sequential if and only if there is $\theta < \omega_1$ such that $A^\theta = \text{cl}_X A$ (where $\text{cl}_X A$ denotes the closure of A in X) for all $A \subseteq X$. The minimal ordinal θ with this property is called the *sequential order* of a sequential space X and will be denoted by $\sigma(X)$. For a sequential space X and $x \in X$, we define

$$\sigma(x, X) = \min\{\theta \leq \omega_1 : \forall A \in \mathcal{P}(X)(x \in \text{cl}_X A \Rightarrow x \in A^\theta)\}.$$

If S is a sequential space that contains $\xi(\mathcal{F})$, then the sequential order of \mathcal{F} inside the space S is the ordinal number

$$\sigma(\mathcal{F}, S) = \min\{\theta \leq \omega_1 : \forall A \# \mathcal{F} (\mathcal{F} \in A^\theta)\},$$

where the iteration A^θ is taken inside of the space S . The *subsequential order* of a subsequential filter \mathcal{F} is the ordinal number

$$\sigma(\mathcal{F}) = \min \{ \sigma(\mathcal{F}, S) : S \text{ is a sequential space with } \xi(\mathcal{F}) \subseteq S \}.$$

In the mentioned paper authors proved the following result:

Theorem 2.4 ([10, Theorem 3.2 in notation of this paper]). *Let $(A_n)_{n < \omega}$ be a partition of ω into infinite sets. Let \mathcal{F}_n be a subsequential filter on A_n and let \mathcal{F} be a subsequential filter on ω . Then $\sigma(\int_{\mathcal{F}} \mathcal{F}_n) \leq \min \{ \sup \{ \sigma(\mathcal{F}_n) : n \geq m \} : m < \omega \} + \sigma(\mathcal{F})$.*

and asked weather inequivalence can be replaced by equivalence, formally:

Question 2.5 ([10, Question 3.3 in notation of this paper]). *Let $(A_n)_{n < \omega}$ be a partition of ω into infinite sets. If \mathcal{F} is a subsequential filter on ω and \mathcal{F}_n is a subsequential filter on A_n , is it true that $\sigma(\int_{\mathcal{F}} \mathcal{F}_n) = \min \{ \sup \{ \sigma(\mathcal{F}_n) : n \geq m \} : m < \omega \} + \sigma(\mathcal{F})$?*

We answer the question negatively (see Examples 2.8, 2.9, 2.10 and Theorem 2.15). Moreover, we find more precise estimation of $\sigma(\int_{\mathcal{F}} \mathcal{F}_n)$ (Theorem 2.6) and show that it is still not the best possible (see Examples 2.8, 2.9 and Theorem 2.15), but first let us define:

Let $(A_n)_{n < \omega}$ be a partition of ω into infinite sets. Let \mathcal{F}_n be a subsequential filter on A_n and let \mathcal{F} be a subsequential filter on ω . Then

$$\sigma((\mathcal{F}_n) \text{ mod } \mathcal{F}) \text{ is by definition } \min \{ \sup \{ \sigma(\mathcal{F}_n) : n \in F \} : F \in \mathcal{F} \}.$$

Theorem 2.6. *Let $(A_n)_{n < \omega}$ be a partition of ω into infinite sets. Let \mathcal{F}_n be a subsequential filter on A_n and let \mathcal{F} be a subsequential filter on ω . Then $\sigma(\int_{\mathcal{F}} \mathcal{F}_n) \leq \sigma((\mathcal{F}_n) \text{ mod } \mathcal{F}) + \sigma(\mathcal{F})$.*

Proof. We proceed similarly to the proof of [10, Theorem 3.2]. Take S_0 such that $\xi(\mathcal{F}) \subset S_0 = \text{cl}_{S_0} \omega$, $\sigma(\mathcal{F}, S_0) = \sigma(\mathcal{F})$ and S_n such that $\xi(\mathcal{F}_n) \subset S_n = \text{cl}_{S_n} A_n$, $\sigma(\mathcal{F}_n, S_n) = \sigma(\mathcal{F}_n)$. Without loss of generality we may assume that $S_i \cap S_j = \emptyset$ for $0 \leq i < j$. Let S be a quotient space obtained by identification of $\mathcal{F}_n \in S_n$ with $n \in S_0$. Such a space is sequential by [8]. For $F \in \mathcal{F}$ define $F^\uparrow = \bigcup_{n \in F} S_n$. Let $A \# \int_{\mathcal{F}} \mathcal{F}_n$ and let $F \in \mathcal{F}$ be such that $\sigma((\mathcal{F}_n) \text{ mod } \mathcal{F}) = \sup \{ \sigma(\mathcal{F}_n) : n \in F \}$. There exists an open neighbourhood \tilde{F} of $\int_{\mathcal{F}} \mathcal{F}_n$ (in S) such that $F^\uparrow \subset \tilde{F}$ and $(F^\uparrow)^c \cap \tilde{F} = \emptyset$. Split A into $A \cap F^\uparrow$ and $A \cap (F^\uparrow)^c$. Then $\int_{\mathcal{F}} \mathcal{F}_n \notin \text{cl}_S (A \cap (F^\uparrow)^c)$ and $\int_{\mathcal{F}} \mathcal{F}_n \in \text{cl}_S (A \cap F^\uparrow)$. Moreover, if $\int_{\mathcal{F}} \mathcal{F}_n \in A^\theta$, then $\int_{\mathcal{F}} \mathcal{F}_n \in (A \cap F^\uparrow)^{\theta_1}$ for some $\theta_1 \geq \theta$. By choice of S_n for each $n \in F \subset S$ there is $\mathcal{F}_n \in (A \cap F^\uparrow)^{\sigma(\mathcal{F}_n)}$ and also $\mathcal{F} \in F^{\sigma(\mathcal{F})}$. Thus $\int_{\mathcal{F}} \mathcal{F}_n \in (A \cap F^\uparrow)^{\sup \{ \sigma(\mathcal{F}_n) : n \in F \} + \sigma(\mathcal{F})}$, so $\sigma(\int_{\mathcal{F}} \mathcal{F}_n, S) \leq \sup \{ \sigma(\mathcal{F}_n) : n \in F \} + \sigma(\mathcal{F})$ and so $\sigma(\int_{\mathcal{F}} \mathcal{F}_n, S) \leq \sigma((\mathcal{F}_n) \text{ mod } \mathcal{F}) + \sigma(\mathcal{F})$, therefore $\sigma(\int_{\mathcal{F}} \mathcal{F}_n) \leq \sigma((\mathcal{F}_n) \text{ mod } \mathcal{F}) + \sigma(\mathcal{F})$. \square

Although the above Theorem 2.6 extends Theorem 2.4 (see Example 2.10), we cannot replace the inequality with equality, what may be seen in the following Examples 2.8 for successor $\sigma(\mathcal{F})$ and 2.9 for limit $\sigma(\mathcal{F})$, which answer Question 2.5.

To describe examples we need to recall definition (from [10]) of the power of Fréchet filter. To this end we need an auxiliary operation: Fix filter \mathcal{F} on ω and a partition $\{A_n : n < \omega\}$ of ω into infinite subsets. For each $n < \omega$, let \mathcal{F}_n be a filter on A_n . Then

we define $\prod \mathcal{F}_n = \{\bigcup_{n < \omega} F_n : \forall n < \omega (F_n \in \mathcal{F}_n)\}$. Note that $\prod \mathcal{F}_n = \int_{\{\omega\}} \mathcal{F}_n$, where $\{\omega\}$ is a principal filer. Additionally for each ordinal $\theta < \omega_1$ we fix strictly increasing unbounded sequence $\{\theta_n : n < \omega\}$ in θ consisting of no limit ordinal numbers. Let $\mathcal{F}_r^1 = \mathcal{F}_r$,

- if $\theta = \mu + 1$ and μ is a successor, then $\mathcal{F}_r^\theta = \int_{\mathcal{F}_r} \mathcal{F}_n$, where \mathcal{F}_n are copies (by bijection) of \mathcal{F}_r^μ on A_n ;
- if θ is a limit, then $\mathcal{F}_r^\theta = \prod \mathcal{F}_n$ and $\mathcal{F}_r^{\theta+1} = \int_{\mathcal{F}_r} \mathcal{F}_n$ where \mathcal{F}_n are copies (by bijection) of $\mathcal{F}_r^{\theta_n}$ on A_n , note also that by LFPP (see section “Almost the same cascades”) considering $\mathcal{F}_r^{\theta+1}$ for limit θ it is sufficient for the sequence mentioned above to be nondecreasing (instead of increasing). Note also that for each successor countable θ there is a monotone sequential cascade V such that $\mathcal{F}_r^\theta = \int V$ and
 - 1) $\theta = r(V)$ for finite θ ;
 - 2) $\theta = r(V) + 1$ for infinite θ .

To have the inverse situation as well, we would have to allow different sequence of ordinals that converges to a limit ordinal number in the definition.

Theorem 2.7 ([11, Theorem 4.8]). $\sigma(\mathcal{F}_r^\alpha) = \alpha$.

Example 2.8. Let $(A_n)_{n < \omega}$ and $(B_k)_{k < \omega}$ be partitions of ω into infinite sets. Let $\mathcal{V}_n = \mathcal{F}_r(A_n)$ for even n and $\mathcal{V}_n = \mathcal{F}_r^2(A_n)$ for odd n and let $\mathcal{F} = \int_{\mathcal{F}_r} \mathcal{V}_n$. Define $\mathcal{F}_k = \mathcal{F}_r^2(B_k)$ for $k \in A_n$ for even n and $\mathcal{F}_k = \mathcal{F}_r(B_k)$ for $k \in A_n$ for odd n . Then $\int_{\mathcal{F}} \mathcal{F}_k = \mathcal{F}_r^4$ for some partition and $\sigma(\int_{\mathcal{F}} \mathcal{F}_k) = 4$ by Theorem 2.7, while the estimation of $\sigma(\int_{\mathcal{F}} \mathcal{F}_k)$ by Theorem 2.4 and by Theorem 2.6 is 5.

Example 2.9. Let $(A_n)_{n < \omega}$ and $(B_k)_{k < \omega}$ be partitions of ω into infinite sets. Let $\mathcal{F} = \int_{\mathcal{F}_r} \mathcal{F}_r(A_n)$. Define $\mathcal{F}_k = \mathcal{F}_r^k(B_k)$ for $k \in A_n$. Then $\int_{\mathcal{F}} \mathcal{F}_k = \mathcal{F}_r^{\omega+1}$ and by Theorem 2.7 $\sigma(\int_{\mathcal{F}} \mathcal{F}_k) = \omega + 1$, while the estimation of $\sigma(\int_{\mathcal{F}} \mathcal{F}_k)$ by Theorem 2.4 and by Theorem 2.6 is $\omega + 2$.

Example 2.10. Let $(A_n)_{n < \omega}$ and $(B_k)_{k < \omega}$ be partitions of ω into infinite sets. Let $\mathcal{F} = \int_{\mathcal{F}_r} \mathcal{F}_n(A_n)$ and let $k(n)$ be a fixed point in A_n . Let $\mathcal{F} = \int_{\mathcal{F}} \mathcal{F}_r(A_n)$, $\mathcal{F}_k = \mathcal{F}_r(B_k)$ for $k \in A_n$, $k \neq k(n)$ and $\mathcal{F}_{k(n)} = \mathcal{F}_r^2(B_{k(n)})$. Then the estimation of $\sigma(\int_{\mathcal{F}} \mathcal{F}_k)$ by Theorem 2.4 is 4, while by Theorem 2.4 is 3, and this is the correct subsequential order of $\int_{\mathcal{F}} \mathcal{F}_k$.

Above examples may suggest that $\sigma(\int_{\mathcal{F}} \mathcal{F}_n) = \min \{\sup \{\sigma(\mathcal{F}_n) + \sigma(\mathcal{F}), : n \in F\} : F \in \mathcal{F}\}$, but it is not the case.

Example 2.11. Let $(A_n)_{n < \omega}$ and $(B_k)_{k < \omega}$ be partitions of ω into infinite sets. For each $n < \omega$ define $f_n : A_n \rightarrow \omega$ as an arbitrary fixed bijection. Let $\mathcal{F}_k = \mathcal{F}_r^{f_n(k)}(B_k)$ for $k \in A_n$. Then $(\int_{\mathcal{F}_r} \mathcal{F}_k) = \mathcal{F}_r^{\omega+1}$, and so by Theorem 2.7 $\sigma(\int_{\mathcal{F}} \mathcal{F}_k) = \omega + 1$, while by formula above it is ω .

We finish this section by stating a theorem showing how independent is $\sigma(\int_{\mathcal{F}} \mathcal{F}_n)$ from $\sigma((\mathcal{F}_n \text{ mod } \mathcal{F}))$ and from $\sigma(\mathcal{F})$. To prove it we need three remarks, which proofs are easy to check.

Remark 2.12. Let α be a successor countable ordinal and β a countable ordinal. Then $\int_{\mathcal{F}_r^\beta} \mathcal{F}_n = \mathcal{F}_r^{\alpha+\beta}$, where (\mathcal{F}_n) are \mathcal{F}_r^α on distinct elements of some partition of ω .

Remark 2.13. Let $(A_n)_{n < m}$ be a finite(!) partition of ω and let \mathcal{F} be a subsequential filter such that $A_n \# \mathcal{F}$ for each $n < m$. Then $\sigma(\mathcal{F}) = \max \{ \sigma(\mathcal{F} \upharpoonright_{A_n}) : n < m \}$.

Remark 2.14. (Folklore) Under assumptions of Theorem 2.15, there are γ_1, γ_2 such that $1 \leq \gamma_1 \leq \alpha, 1 \leq \gamma_2 \leq \beta$ and $\gamma_1 + \gamma_2 = \gamma$.

Theorem 2.15. Let α, β be countable ordinals, let $\delta(\alpha, \beta) = \alpha + \beta$ if α is a successor, $\delta(\alpha, \beta) = \alpha + (-1 + \beta)$ if α is a limit and let $\gamma \in \{ \min \{ \alpha + 1, 1 + \beta \}, \dots, \alpha + \beta \}$. Then there is a discrete sequence $(\mathcal{F}_n)_{n < \omega}$ of subsequential filters and a subsequential filter \mathcal{F} such that $\sigma(\int_{\mathcal{F}} \mathcal{F}_n) = \gamma$, while $\sigma((\mathcal{F}_n) \bmod \mathcal{F}) = \alpha$ and $\sigma(\mathcal{F}) = \beta$.

Proof. Consider partitions $(A_n)_{n < \omega}, (B_n)_{n < \omega}$ of ω into infinite sets.

Take $\hat{\gamma} = \gamma$ for $\gamma < \omega$ and $\hat{\gamma} = \gamma - 1$ for infinite successor γ and $\hat{\gamma} = \gamma$ for limit γ . Consider γ_1, γ_2 from Remark 2.14 used for $\alpha, \beta, \hat{\gamma}$.

We have two possibilities:

- 1) There is a pair γ_1, γ_2 , such that γ_1 is a successor, note that if α is a successor we are always in this case;
- 2) For each pair γ_1, γ_2 , number γ_1 is a limit.

Case 1:

Let $(\theta_i)_{i < \omega}, (\beta_i)_{i < \omega}$ be nondecreasing sequences of successor ordinals such that $(\theta_i + 1) \rightarrow \gamma_2$ and $(\beta_i + 1) \rightarrow \beta$.

For each $n < \omega$ define \mathcal{V}_i as filters on A_i such that:

$$\mathcal{V}_i = \mathcal{F}_r^{\beta_i} \text{ for } i \in \{3n : n < \omega\};$$

$$\mathcal{V}_i = \mathcal{F}_r^{\theta_i} \text{ for } i \in \{3n + 1 : n < \omega\};$$

$$\mathcal{V}_i \text{ is a principal ultrafilter for } i \in \{3n + 2 : n < \omega\}.$$

Put $\mathcal{V} = \mathcal{F}_r$ for successor $\gamma_2, \mathcal{V} = \{\omega\}$ for limit γ_2 and define $\mathcal{F} = \int_{\mathcal{V}} \mathcal{V}_i$.

Let \mathcal{F}_k be a filter on B_k such that

$$\mathcal{F}_k = \mathcal{F}_r^1 \text{ for } k \in A_{3n} \text{ for some } n;$$

$$\mathcal{F}_k = \mathcal{F}_r^{\gamma_1} \text{ for } k \in A_{3n+1} \text{ for some } n < \omega;$$

$$\mathcal{F}_k = \mathcal{F}_r^\alpha \text{ for } k \in A_{3n+2} \text{ for some } n.$$

Denote $\mathcal{T} = \int_{\mathcal{F}} \mathcal{F}_k$ and consider sets $C_1 = \bigcup \{ B_k : k \in A_{3n}, n < \omega \}, C_2 = \bigcup \{ B_k : k \in A_{3n+1}, n < \omega \}, C_3 = \bigcup \{ B_k : k \in A_{3n+2}, n < \omega \}$.

Note that by Remark 2.12 $\mathcal{T} \upharpoonright_{C_1} = \mathcal{F}_r^\gamma, \mathcal{T} \upharpoonright_{C_2} = \mathcal{F}_r^{1+\beta}, \mathcal{T} \upharpoonright_{C_3} = \mathcal{F}_r^{\alpha+1}$. Thus, by Remark 2.13 and by Theorem 2.7, $\sigma(\mathcal{T}) = \gamma$. Computation of $\sigma((\mathcal{F}_n) \bmod \mathcal{F})$ and $\sigma(\mathcal{F})$ is obvious.

Case 2: γ_1 is a limit.

We work similiary to case 1, but this time we need to use cascades (instead of their contours) and exploit their structure.

Let $(\theta_i)_{i < \omega}, (\beta_i)_{i < \omega}$ and $(\mu_i)_{i < \omega}$ be nondecreasing sequences of successor ordinals such that $(-1 + \theta_i + 1) \rightarrow \gamma_2, (\beta_i + 1) \rightarrow \beta$ and $(\mu_i) \rightarrow \gamma_1$.

For each $n < \omega$ define V_i as a monotone sequential cascade such that $\max V_i = A_i, r(v^-) = 1$ for all $v \in \max V_i$ for all $i < \omega$ such that

$$r(V_i) = \beta_i \text{ for } i \in \{3n : n < \omega\};$$

$$r(V_i) = \theta_i \text{ for } i \in \{3n + 1 : n < \omega\};$$

$r(V_i) = 0$ for $i \in \{3n + 2 : n < \omega\}$.

Put $\mathcal{V} = \mathcal{F}_r$ for successor γ_2 , $\mathcal{V} = \{\omega\}$ for limit γ_2 and define $\mathcal{F} = \int_{\mathcal{V}} \int V_i$.

Let \mathcal{F}_k be a filter on B_k such that

$\mathcal{F}_k = \mathcal{F}_r^1$ for $k \in V_{3n}$ for some n ;

$\mathcal{F}_k = \mathcal{F}_r^{\mu_i}$ for $k = v_i, v \in V_{3n+1}$ for some $n < \omega$ and $r(v) = 1$;

$\mathcal{F}_k = \mathcal{F}_r^\alpha$ for $k \in V_{3n+2}$ for some n .

The rest of the proof follows like in the case 1. □

2.2. Subcascades defined by subsets of $\max V$

First way of finding subcascades is to do it by node. Let V be a cascade and let $v_0 \in V$. Then V^\uparrow is $\{v \in V : v \sqsubset v_0\}$ with unchanged order, so it is extremely natural and easy to use.

In the second method for any set $U \subset \omega$ we define $V^{\downarrow U}$ as follows (recall that $\max V \subset \omega$). If $U \# \int V$, then we set $V^{\downarrow U} = \emptyset$. Assume that $U \# \int V$. If $r(V) = 0$, then $V^{\downarrow U} = V$. If $r(V) = 1$, then $V^{\downarrow U} = \{\emptyset_V\} \cup \{v \in \max V : v \in U\}$. If $r(V) \geq 2$, then $V^{\downarrow U} = \{\emptyset_V\} \cup \bigcup V_n^{\downarrow U}$. Note that if $U \# \int V$, then $V^{\downarrow U}$ preserves sequentiality, monotonicity of V and the rank of monotone sequential cascade V . Note also, that $V^{\downarrow U}$ may be different from $V^{\downarrow\downarrow U} = \{v \in V : v^\uparrow \cap U \neq \emptyset\}$.

For a comfortable use of the restriction of a cascade by set we will sometimes need a lemma from Section “Almost the same cascades”, which shows for example for a fixed monotone sequential cascade V and a set $A \# V$ that we can “slightly” change a cascade to obtain a cascade with the same values of notion of interest for us and such that $\max V^{\downarrow A} = A$, and so an example of use is in section “Almost the same cascades”.

2.3. Two types of inversed images

Clearly we could define more types, but two presented here are usually used in proofs. Take a monotone sequential cascade V and a function $f : \omega \rightarrow \omega$, $\max V \subset f[\omega]$. To define the first type of inversed images we additionally assume that

$$r(v) > 1 \text{ implies } \text{card}(f^{-1}[v^+ \cap \max V]) < \infty.$$

This condition guarantees that inversed image of a monotone sequential cascade is again a monotone sequential cascade, more precisely it guarantees monotonicity. If the condition is not fulfilled, then we may consider $\tilde{V} = V \setminus A$, where $A = \{v \in \max V : r(v^- \geq 2), \text{card}(f^{-1}(v)) = \infty\}$. Note that \tilde{V} is a monotone sequential cascade and that rank, contour and order are the same as in V .

We define $f^{-1}(V) = (V \setminus \max V) \cup f^{-1}[\max V]$, with an order described below. First define a function $g : \omega \cup V \rightarrow \omega \cup V$, by $g(n) = f(n)$ for $n \in \omega$, $g(v) = v$ for $v \notin \omega$. Now $v' \sqsubseteq_{\tilde{V}} v''$ if and only if $g(v') \sqsubseteq_V g(v'')$. We may also leave lexicographic order almost unchanged, i.e., unchanged in elements of rank ≥ 2 .

This type of inversed limit is used for example in the proof of

Theorem 2.16 ([20, Theorem 2.3]). *If $f : \omega \rightarrow \omega$ and $u \in \mathcal{P}_\alpha$, then $f(u) \in \mathcal{P}_\beta$ for a certain $\beta \leq \alpha$.*

Simply, without loss of generality, assume that $\alpha < \omega_1$ and assume that the claim does not hold. Take a cascade V of rank $\gamma \geq \alpha$ such that $\int V \subset f(u)$ and prove that $\int(f^{-1}(V)) \subset u$.

The second type of inversed images makes sense if there is a “big” set in an image such that inversed images of elements of this set are infinite. Here “big” may mean “belonging to $\int V$ ”, or “that meshes $\int V$ ”, or “that belongs to the (predetermined) (ultra)filter that contain $\int V$ ”, or ... In two last cases we usually may restrict the cascade to the set of infinite inversed images (sometimes using same lemmas that “almost do not change” the cascade), thus we define it in the simplest case when $\max V$ is contained in the mentioned set.

Let $f : \omega \rightarrow \omega$ be a function such that $\text{card}(f^{-1}(v)) = \infty$ for each $v \in \max V$. Here this extra condition guarantees sequentiality of an inversed image of a sequential cascade. Formally we have to assume that range and image of f are in the disjoint copies of ω . Define $f^{-1}(V) = V \cup f^{-1}[\max V]$, the order on $f^{-1}(V)$ is an extension by transitivity of the following one: on $V \subset f^{-1}(V)$ we keep an order of V and $n \sqsubset_{f^{-1}(V)} f(n)$ for $n \in f^{-1}(\max V)$. In another words, under above conditions, we may say that for a monotone sequential cascade V of rank 0, i.e., for a point, $f^{-1}(V)$ is a monotone sequential cascade of rank 1: $f^{-1}(V) = V \cup f^{-1}[V]$, where $\emptyset_{f^{-1}(V)} = V$. For a cascade V of rank ≥ 1 define $f^{-1}(V) = V \uparrow_{v \in \max V} f^{-1}(v^\uparrow)$.

We used this type of inversed image for example to show correspondence between cascades and $<_\infty$ sequences, what lets us answer questions from [16].

Given ultrafilters u, v on ω , recall that $v <_\infty u$ if there is a function $f : \omega \rightarrow \omega$ such that $f(u) = v$ and f is neither finite-to-one nor constant on each set $U \in u$. Laflamme proved that if an ultrafilter u has an infinite decreasing $<_\infty$ -sequence below, then u is at least strict $J_{\omega^{\omega+1}}$ -ultrafilter (c.f.[16, Lemma 3.2]). He also stated the following

Problem 2.17. *What about the corresponding influence of increasing $<_\infty$ -chains below u ? Given such an ultrafilter u with an increasing infinite $<_\infty$ -sequence $u >_{RK} \dots >_\infty u_1 >_\infty u_0$ below, fix maps g_i and f_i witnessing $u >_{RK} u_i$ and $u_{i+1} >_\infty u_i$ respectively. The problem is really about the possible connections between g_i and $f_i \circ g_{i+1}$ even relative to members of u .*

Problem 2.18. *Can we have an ultrafilter u with arbitrary long finite $<_\infty$ -chains below u without infinite one? This looks like the most promising way to build a strict J_{ω^ω} -ultrafilter.*

We find an affirmative answer to the first problem and a negative answer to the second one. For details see [22].

So how does this correspondence between cascades and $<_\infty$ -sequences of ultrafilters look like:

Let u be an ultrafilter, suppose that there is a sequence $u = u_0 >_\infty u_1 >_\infty \dots >_\infty u_n$ of ultrafilters and take functions $f_m : \omega \rightarrow \omega$, each witnessing that $u_{m-1} >_\infty u_m$. Since formally levels in a cascade cannot intersect, we may assume that domain of f_1 and ranges of f_m are subsets of pairwise disjoint copies of ω .

We will build a monotone sequential cascade V which corresponds to the sequence above with respect to some $U \in u$. Simply start with a monotone sequential cascade of rank 1 and on each of n steps take an inversed image of the cascade (by f_{n-k+1} for step k). Those inverse images may not be sequential, but a set of elements of infinite inversed images belongs to the u_{n-k+1} (for step k), so we can restrict our cascade and make the next step on this restricted cascade. The number of restrictions is finite and at the end we obtain the cascade we were looking for.

Now take any monotone sequential cascade V of finite rank, with $\int V \subset u$. Without loss of generality we may assume that all branches of V have the same length n . For each $v \in V$ let \hat{v} be an arbitrary element of $\max v^\uparrow$. Consider functions $f_i : \omega \rightarrow \omega$ such that $f_i(v_1) = \hat{v}$ for each $v_1 \in \max v^\uparrow$, where $r(v) = i$. Thus $u >_\infty f_1(u) >_\infty f_2(u) >_\infty \dots >_\infty f_n(u)$.

2.4. Decreasing the rank, i.e., destroying nodes

We say that a cascade V is *built by destruction of nodes of rank 1* in a cascade W of rank $r(W) \geq 2$ if and only if $V = W \setminus R$, where $R = \{w \in W : r(w) = 1, r(w^-) = 2\}$.

Observe that if W is a monotone sequential cascade, then V is also a monotone sequential cascade. Moreover, if $r(W)$ is finite, then $r(V) = r(W) - 1$ and if $r(W)$ is infinite, then $r(V) = r(W)$.

Assume that there is a given cascade of rank α and an ordinal $1 \leq \beta \leq \alpha$. We shall describe the operation of *decreasing the rank* of a cascade W to β . The construction is inductive:

For a finite α , we can decrease the rank of W from α to β by applying $\alpha - \beta$ times the operation of destroying nodes of rank 1 (i.e., if $\alpha = \beta$, then the cascade is unchanged).

For infinite α , suppose that for each pair (δ, γ) , where $1 < \delta \leq \gamma < \alpha$, and for each cascade W of rank γ the operation of decreasing the rank of W from γ to δ is defined. Let W be a monotone sequential cascade of rank α , let (β_n) be a nondecreasing sequence of ordinals such that $\beta_n = 0$ if and only if $r(W_n) = 0$, $\beta_n \leq r(W_n)$ and $\lim_{n \rightarrow \infty} (\beta_n + 1) = \beta$. For each $n < \omega$ let V_n be the cascade obtained by decreasing the rank of W_n to β_n . Finally, let $V = (n) \leftrightarrow V_n$.

Clearly, for infinite α the operation of decreasing the rank is not defined uniquely. Observe also that the described above decrease of the rank of a cascade W does not change $\max W$. If a cascade V is obtained from W by decreasing the rank, then we write $V \triangleleft W$. Trivially $V \triangleleft V$ and inductively $V \triangleleft W \Rightarrow \int V \subset \int W$.

This operation has been used for example in the proof of Lemma 2.19. The idea is to take (by contradiction) a sequence of cascades and a cascade of limit rank which contour is in the sum of contours of sequence of cascade. By [6] for each cascade from the sequence there is a set in limit contour which does not belong to the contour of the cascade in sequence. The point is to control “shrinking” of these sets, to be fixed on some parts of limit cascades. To do it we apply the operation described above.

Lemma 2.19 ([21, Lemma 6.3]). *Let $\alpha < \omega_1$ be a limit ordinal and let $(\mathcal{V}^n : n < \omega)$ be a sequence of monotone sequential contours such that $r(\mathcal{V}^n) < r(\mathcal{V}^{n+1}) < \alpha$ for every n and such that $\bigcup_{n < \omega} \mathcal{V}^n$ has the finite intersection property. Then there is no monotone sequential contour \mathcal{W} of rank α such that $\mathcal{W} \subset (\bigcup_{n < \omega} \mathcal{V}^n)$.*

2.5. Almost the same cascades

In this section we quote “trash lemma” and show one operation – “locally finite partition” which also “almost” does not change the cascade.

Let V be a monotone sequential cascade, $A \subset \omega$. We define $\text{unseq}(V, A) = \bigcup\{(\max v^\uparrow) \setminus A : v \in V, (\max v^\uparrow) \setminus A \not\# \int v^\uparrow\}$. If additionally $\mathbb{A} = \{A_s : s \in S\}$ is a family of subsets of ω , then we define $\text{unseq}(V, \mathbb{A}) = \bigcup_{T \subset S} \text{unseq}(V, \bigcup_{s \in T} A_s)$.

Let V and W be monotone sequential cascades such that $\max V \supset \max W$. We say that W *increases the order of* V and we write $W \Rightarrow V$ if $\text{ot}(f_W(U)) \geq \text{indec}(\text{ot}(f_V(U)))$ for each $U \subset \max W$, where $\text{indec}(\alpha)$ is the biggest indecomposable ordinal $\leq \alpha$ (by Cantor normal form theorem such a number exists and is defined uniquely). Clearly this relation is idempotent and transitive.

Lemma 2.20 ([22, Lemma 3.4]). *Let V be a monotone sequential cascade, $(A_i)_{i=0, \dots, m}$ be a partition of $\max V$ such that $A_i \not\# \int V$ for each $i = 1, \dots, m$. Then there is a monotone sequential cascade \tilde{V} such that*

- 1) $\max V = \max \tilde{V}$;
- 2) $V \Leftrightarrow \tilde{V}$;
- 3) $\int V \downarrow C = \int \tilde{V} \downarrow C$ for each $C \# \int V$;
- 4) $V \downarrow\downarrow D = \tilde{V} \downarrow\downarrow D$ for $D = \omega \setminus [\text{unseq}(V, \{\max V \downarrow A_i : i = 1, \dots, m\}) \cup \bigcup_{i=1}^m (A_i \setminus \max V \downarrow A_i)]$;
- 5) $\max \tilde{V} \downarrow A_i = A_i$ for each $i = 1, \dots, m$.

The idea of *locally finite partition* of a cascade is to split the set v^+ for $v \notin \{\emptyset_V\} \cup \max V$ into finitely many infinite pieces, and in the place of v put the same number of copies of v such that a set of new immediate successors of each copy of v is a (different) piece of the old v^+ . Formally:

Let V, W be monotone sequential cascades, $V \subset W$, $\max V = \max W$, $\emptyset_V = \emptyset_W$. If there is a finite-to-one function $f : W \rightarrow V$ such that

- 1) $f \upharpoonright_V = \text{id}_V$;
- 2) $f[w^+] \subset (f(w))^+$ for all $w \in W$;
- 3) $w', w'' \in f^{-1}(v)$ implies $\text{pred}_W w' = \text{pred}_W w''$,¹

then we say that W is obtained from V by *locally finite partition*. In such a situation, $\int V = \int W$ and $W \Leftrightarrow V$. We call these properties of a locally finite partition *locally finite partition properties* (LFPP).

One may think about the locally finite partition (locally – i.e., on each non-ekstremal node) as of the inversed operation of gluing:

Let $(V^i)_{i < \omega}$ be a pairwise disjoint sequence of cascades of ranks ≥ 1 . Then $\bigoplus_{i < \omega} V^i$ is a cascade obtained from $\bigcup_{i < \omega} V^i$ by identification $\emptyset_{V^1} = \emptyset_{V^2} = \dots = \emptyset_{V^n} = \dots$

Where to apply it – it is essentially usefull when we work with ultrafilters that contain contours of cascades. For example we compare elements of antichains in two cascades, thus we may look at pairs of them as on elements of the $\omega \times \omega$ matrix. Since we work with an ultrafilter, either supdiagonal or subdiagonal (by image) belongs

¹ Simply we may say that function f glues disjoint finite subsets of w^+ , for such $w \in W$ that $r(w) > 1$, into elements of this subsets and leaves \emptyset_W and $\max W$ unchanged; V is a result of this gluing.

to the ultrafilter, and now using “trash lemma” we may compare finite-to-infinite or infinite-to-finite elements (it is also possible that there exists a finite-finite subset). Here we apply LFPP, which leads us to the situation 1-to-infinite, infinite-to-1, or 1-1. That can allow us, for instance, to do further induction (by rank).

This technique was used for example in the proof of [22, Theorem 3.11] – the main theorem of a paper and simultaneously a key to the proof of

Theorem 2.21. (ZFC) *The class of strict J_{ω} -ultrafilters is empty.*

which partially answers Baumgartner question from [1] – whether ordinal ultrafilters of limit index may exist (even under some extra set-theoretical assumptions).

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